Math 246A Lecture 8 Notes

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1 Cauchy's Integral Formula and Morera's Theorem

1.1 Cauchy's integral formula for rectangles

Let $\Omega \subseteq \mathbb{C}$ be a domain. Last time we proved the following.

Lemma 1.1. *If R is a rectangle with $\overline{R} \subseteq \Omega$ and $f \in H(\Omega)$, then*

$$\int_{\partial R} f(z)\,dz = 0.$$

Let's go further with this result.

Lemma 1.2. *Let R be a rectangle. Then*

$$\int_{\partial R} \frac{1}{z-a}\,dz = \begin{cases} 0 & a \notin R \\ 2\pi i & a \in R. \end{cases}$$

Proof. If $a \notin \overline{R}$, use the previous lemma. Other wise, let $S = \{x : |x - \operatorname{Re}(a)| < \delta\} \cup \{y : |y - \operatorname{Im}(a)| < \delta\}$ be a square. Then $\overline{S} \subseteq R$. Split the rectangle R into disjoint rectangles including S, where the only rectangle containing a is S. Change the contour appropriately.

The contributions of the other rectangles to the contour integral all are zero by the previous lemma. So

$$\int_{\partial R} \frac{1}{z-a}\,dz = \int_{\partial S} \frac{1}{z-a}\,dz.$$

Without loss of generality, $\delta = 1$ and $a = 0$. Then this is the integral

$$= \underbrace{\int_{-1}^{1} \frac{1}{x-i}\,dx}_{I} + \underbrace{\int_{-1}^{1} \frac{i}{1+iy}\,dy}_{II} - \underbrace{\int_{-1}^{1} \frac{1}{x+i}\,dx}_{III} + \underbrace{\int_{-1}^{1} \frac{i}{-1+iy}\,dy}_{IV}$$

Note that

$$I + III = \int_{-1}^{1} \frac{1}{x-i} - \frac{1}{x+i}\,dx = 2i \int_{-1}^{1} \frac{1}{1+x^2}\,dx = 2i(\tan^{-1}(1) - \tan^{-1}(-1)).$$

You can find that $II + IV$ is equal to the same thing. $\square$

Theorem 1.1 (Cauchy integral formula for rectangles)**.** *Let $\overline{R} \subseteq \Omega$ and $f \in H(\Omega)$. Then*

$$\frac{1}{2\pi i}\int_{\partial R} \frac{f(z)}{z-a}\, dz = \begin{cases} 0 & a \notin \overline{R} \\ f(a) & a \in \operatorname{int}(R). \end{cases}$$

Proof. If $a \in R$, then this is

$$\frac{1}{2\pi i}\int_{\partial S} \frac{f(z)}{z-a}\, dz,$$

where S is a square. The previous lemma gives us that

$$\left|\frac{1}{2\pi i}\int_{\partial S} \frac{f(z)}{z-a}\, dz - f(a)\right| = \left|\frac{1}{2\pi i}\int_{\partial S} \frac{f(z)-f(a)}{z-a}\, dz\right|.$$

Now note that since $f(z) = f(a) + f'(z)(z-a) + o(|z-a|)$,

$$\frac{f(z)-f(a)}{z-a} \xrightarrow{z\to a} f'(a).$$

So as we make the square S smaller, this goes to 0. □

Corollary 1.1. *$H(\Omega) = \mathcal{A}(\Omega)$, where $\mathcal{A}(\Omega)$ is the set of functions $f : \Omega \to \mathbb{C}$ such that f has a convergent power series in some radius around every point in Ω.*

Proof. Let $f \in H(\Omega)$, $z_0 \in \Omega$, $\delta > 0$, and $B(z_0, 2\sqrt{2}\delta) = \{z : |z - z_0| < 2\sqrt{2}\delta\} \subseteq \Omega$.

Let $S \subseteq B(z_0, 2\sqrt{2}\delta)$ be a square around z_0. If $|z - z_0| < \delta$, then

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\partial S} \frac{f(\zeta)}{\zeta - z}\, d\zeta \\ &= \frac{1}{2\pi i}\int_{\partial S} \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)}\, d\zeta \\ &= \frac{1}{2\pi i}\int_{\partial S} \frac{f(\zeta)}{\zeta - z_0}\left(\frac{1}{1 - (z-z_0)/(\zeta - z_0)}\right) d\zeta \end{aligned}$$

The part in the parentheses is $\sum_{n=0}^\infty (z-z_0)^n/(\zeta - z_0)^n$, which is a convergent geometric series.

$$= \sum_{n=0}^\infty \left(\frac{1}{2\pi i}\int_{\partial S} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta\right)(z - z_0)^n.$$ □

1.2 Morera's theorem

Theorem 1.2 (Morera)**.** *Suppose $f : \Omega \to \mathbb{C}$ is continuous, and for all $z_0 \in \Omega$, let $\delta(z_0) > 0$ such that $B(z_0, \delta(z_0)) \subseteq \Omega$. Let $R \subseteq B(z_0, \delta(z_0))$ be a rectangle with sides parallel to the axes, and suppose that*

$$\int_{\partial R} f(z)\, dz = 0.$$

Then $f \in H(\Omega)$.

Proof. Without loss of generality, Ω is a disc $B = \{z : |z - z_0| < a\}$. If $z \in V$, let $\gamma_{z_0,z}$ be a curve joining $z_0 = z(0)$ to $z = z(1)$ consisting of two sides of the rectangle with opposite vertices z_0 and z. Let

$$F(z) = \int_{\gamma_{z_0,z}} f(\zeta)\, d\zeta.$$

F is well-defined because the hypothesis says that this integral is the same no matter which curve we take $\gamma_{z_0,z}$ to be.

Let $z, w \in B$ with $|w - z|$ small. Note that

$$\begin{aligned} |F(w) - F(z) - f(z)(w - z)| &= \left| \int_{\gamma w,z} f(\zeta) - f(w)\, d\zeta \right| \\ &\leq \sup_{|\zeta - w| < |z - w|} |f(\zeta) - f(w)| \cdot |z - w| \\ &= |z - w| \cdot o(|z - w|). \end{aligned}$$

So $F'(u) = f(u)$. Then, since holomorphic implies analytic (we will prove this later), we get that f is holomorphic. $\square$

Next time, we will prove the following.

Theorem 1.3 (Goursat)**.** *Let $f : \Omega \to \mathbb{C}$ be such that $f'(z)$ exists for all $z \in \Omega$. Then $f \in H(\Omega)$.*

Corollary 1.2. *Let $f : \Omega \subseteq \mathbb{C}$. The following are equivalent:*

1. *$f'(z)$ exists for all $z \in \Omega$*
2. *$f \in H(\Omega)$*
3. *$f \in \mathcal{A}(\Omega)$*
4. *$\int_{\partial R} f(z)\, dz = 0$ for all rectangles R with $\overline{R} \subseteq \Omega$.*
5. *f is differentiable, and the matrix $df = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$ satisfies the Cauchy-Riemann equations, $u_x = v_y$, $v_x = -u_y$.*